Appendix S1 Euler's polyhedral's formula and the mean number of neighbors of a cell.

This is a rather heuristic discussion about finite polygonal tilings of the plane and the average number of sides of the tiles.

For any convex polyhedron, we have the famous **Euler's polyhedral formula**:

$$V - E + F = 2$$

where F is the number of faces, E the number of edges between the faces, and V their number of vertices of the polyhedron.

Following Cauchy, we can "look through one face" of the polyhedron and, in so doing, project vertices, edges and faces into a tiling of the plane, where all the faces are bounded, except the projection of the one we're looking through. The figure we obtain in the plane is constituted by a convex polygon tiled itself with convex polygons. Euler's polyhedral formula remains true for this figure, if we continue counting the face that we're looking through, which becomes the infinite face surrounding the polygonal region. It is in fact true for somewhat more general planar graphs [1].

Assuming we have such a tiling of convex polygons, if we denote by p the average number of neighbors each face has then $p = (1/F)\sum n_i$, where n_i is the number of neighbors of Face i. In other words the sum of the number of neighbors for each face is $pF = \sum n_i$. For each face, the number of edges, the number of vertices and the number of neighbors are all equal. By construction each edge is shared by 2 faces, so in the sum of numbers of neighbors, each edge is counted twice: $pF = 2E$. If we further assume (temporarily) that each vertex is of degree 3 then in the sum of number of neighbors, each vertex is counted three times: $pF = 3V$. Replacing these last two expressions in Euler's polyhedral formula, one then obtains $pF/3 - pF/2 + F = 2$ which, solving for p yields

$$p = 6(1 - 2/F).$$

For a large tiling (*i.e.* for large F), one thus has an average of slightly less than 6 neighbors. See [2], p. 65 for a similar fact in the context of Voronoï meshes.

We assume we have such a tilling of convex faces, with edges separating different faces, starting and ending at vertices of degree at least 3 (no edge inside a face, or vertex of degree 1 or 2). The degree of a vertex must "generically" be 3. To see this, we assume here for simplicity that the tilings we're considering are by Voronoï cells, or as described in this article, deformations of such tilings that preserve their original topology.[1] If we allow the smallest random perturbation of the centers of the cells, any vertex of degree 4 (with probability 1) splits into 2 vertices of degree 3. See Fig. S1 for an example. Likewise, under perturbation, vertices of degree 5 generically split into 3 vertices of degree 3, and vertices of degree n split into $n-2$ vertices of degree 3. In other words such higher degree vertices

[1] This fact must be valid for a much larger class of tilings).

are not topologically stable, and generically (*i.e.* with probability 1) our tilings will have only vertices of degree 3.

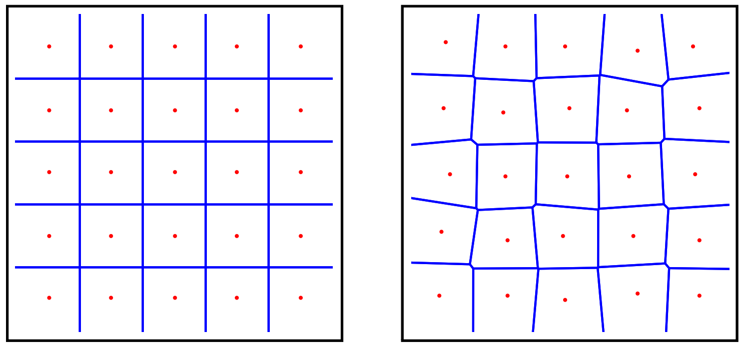

Figure S1: **A Voronoï tiling and its perturbation** On the left, the Voronoï mesh of a perfectly spaced lattice of 25 points. Each vertex has degree 4. On the right, the lattice has been randomly perturbed. Every vertex of the Voronoï polygonal cells has degree 3: each vertex of degree 4 of the lattice's Voronoï mesh has split into two vertices of degree 3 (some are very close to one another. Each vertex contributes a new side to each incident face with probability 1/2.

The contribution of higher degree vertices. Under certain constraints, it might be unavoidable to have vertices of higher degree, however. One can in this case evaluate the contribution of higher degree vertices to the average number of neighbors p.

If the tiling only has vertices of degree 4 (as in the Voronoï mesh for the lattice in Fig. S1), in summing the number of neighbors, each vertex is counted 4 times now, so $pF = 4V$. In general, for a tiling with only degree k vertices, one would obtain $pF = kV$

If the tiling has vertices of mixed degrees (but higher than 2, as we have assumed), we can partition the set of vertices in subsets of vertices of same degree k, and denote the number of such vertices V_k. As seen above, the contribution of vertices of degree k to the sum of neighbors is kV_k, yielding at total of

$$pF = 3V_3 + 4V_4 + 5V_5 + ... + nV_n$$

At the same time the total number of vertices is $V = V_3 + V_4 + V_5 + \ldots + V_n$, which enables us to substitute for V_3 in the previous equation:

$$pF = 3V + V_4 + 2V_5 + 3V_6 + ... + (n-3)V_n.$$

(It is not surprising to find in the coefficients of V_k the number of degree 3 vertices a degree k vertex would generically split into). From this equation, we can in turn solve for V and using $pF = 2E$ and Euler's polyhedral formula, we obtain after some algebra:

$$p = 6 + (-12 + 2V_4 + 4V_5 + 6V_6 + ... + 2(n-3)V_n)/F.$$

This gives the deviation from the mean of 6 neighbors due to higher degree vertices.

A more rigorous treatment of this material would spell out exactly the kind of tilings we're considering, and the kind of randomness allowed in our argument about the genericity of degree 3 vertices. The combinatorial structure of edges and vertices at the boundary of the region is also important, and we avoided this discussion somewhat - and in particular the issue of vertices of degree 2, which we have avoided by fiat (see [3]).

References

[1] Malkevitch, J. (Dec 2004 & Jan 2005) Euler's Polyhedral Formula, AMS feature column,
http://www.ams.org/samplings/feature-column/fcarc-eulers-formula
http://www.ams.org/samplings/feature-column/fcarc-eulers-formulaii

[2] Okabe, A., Boots, B. and Sugihara, K. (1992) Spatial Tessellations – Concepts and Applications of Voronoi Diagrams. (Chichester: John Wiley and Sons)

[3] Graustein, W. C. (1931) On the Average Number of Sides of Polygons of a Net. Annals of Mathematics 32(1): 149-153.