APPLICATION OF DIFFERENT MATHEMATICAL FUNCTIONS TO CALCULATE GROWTH COEFFICIENTS GC AND RGC OF WINTER RAPE PLANTS (*BRASSICA NAPUS* L.)

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(Received: December 8, 1994. Accepted: March 4, 1995)

ABSTRACT

The methodology of calculation of new coefficients, characterizing dry matter accumulation is presented. Eight representative mathematical functions were used and the data referring to the increase of dry matter of winter rape plants (*Brassica napus* L.) during the spring-summer vegetation season. Significance of the applied growth models was verified by using statistical parametrical tests. As a result of the analytical or numerical integration, the growth coefficient (GC) and the relative growth coefficient (RGC) were determined for each function. They describe plant growth potential and mean plant dry matter values during the vegetation season respectively. Depending on the type of the mathematical function the growth coefficient (GC) oscillated within the limits from 1564 g day to 1637 g day, and the relative growth coefficient (RGC) values ranged from 18.8 to 19.7 g.

KEY WORDS: plant growth analysis, dry matter accumulation, growth coefficients, winter rape, *Brassica napus* L.

In connection with more and more universal usage of symbolical and numerical calculations, in botanical sciences, several attempts of mathematical modelling of plant growth were done. Mathematical models (functions) of plant growth graphically presented resemble the S shape curve (the so-called sigmoid curve, Fig.1). In general the sigmoid curves precisely circumscribe a dry matter (w) increase of individual organs, individual plants and even of the whole populations during the vegetation season (t). The sigmoid growth curve can be divided into three phases: exponential, linear and ageing. Generally growth processes present in an initial increase of the absolute growth rate, up to reaching the maximum value (what falls in middle of the linear phase), and then show a decrease. The maximum growth rate $(dw/dt)_{max}$ complies with the growth curve inflection point and zero of the acceleration ($d^2w/dt^2 = 0$). The growth is thought to be finished after reaching the maximal dry matter value by plants, which occurs at t_m moment. Very often growth equations include the dry matter asymptotic value (A), i. e. the limit to which the growth curve is approaching:

$$\lim_{t\to\infty} w(t) = A. \qquad (1)$$

The selection of the continuous function, adequate in a physiologic and statistic meaning, affords possibilities for the accurate growth processes analysis; it also can provide comparisons of the different forms or cultivars vegetation course. A magnitude of the marked area, under the growth curve (Fig. 1), is a dependable measure of the plant growth

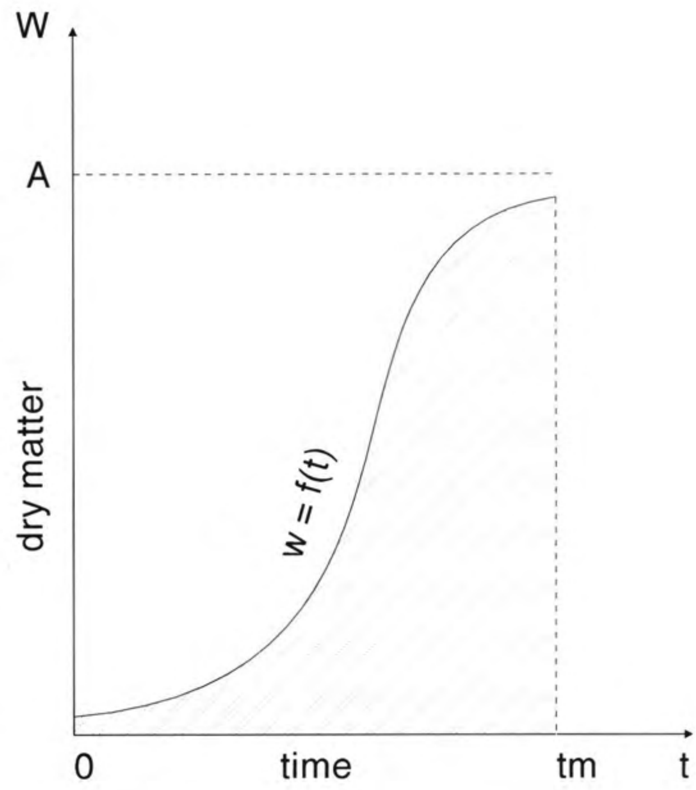

Fig. 1. Graph of the sigmoid growth curve.

potential, for which a name of the growth coefficient (GC) was assumed. That coefficient can be easily calculated as a definite integral of the function w(t) in the limits from 0 to t_m:

$$GC = \int_0^{t_m} w(t)dt. \quad (2)$$

The GC coefficient measure is g day.

A definite growth function integrals calculation affords possibilities for calculation of a plant dry matter mean value ($\overline{w}$) in any interfacial period (t_1, t_2). To this end the mean function value formula should be applied (Radford 1967):

$$\overline{w} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} w(t)dt. \quad (3)$$

If as a lower limit of the integration is replaced by $t_1 = 0$, and $t_2 = t_m$ as an upper, than the mean value of plant dry matter during the whole vegetation season will be received. The calculated in that way parameter is a relative growth coefficient (RGC):

$$RGC = \frac{1}{t_m - 0} \int_0^{t_m} w(t)dt. \quad (4)$$

Its measures expressed in grammes are undoubtedly useful in the plant growth analysis. Simple dependence between GC and RGC coefficients results from the comparison of formula (2) and (4):

$$RGC = \frac{GC}{t_m}. \quad (5)$$

Table 1 presents eight representative growth functions, starting with the simplest linear function. Survey of some of those functions, which have an application in botanical works, can be found in Richards (1969) and also Żelawski's and Lech's (1980) works. Whereas the difference-differential function is the original growth description attempt proposed by Gregorczyk (1986).

The knowledge of growth function integrals is necessary for calculation of the GC and RGC coefficients. Table 1 contains expressions which are indefinite integrals and definite integrals within the limits from 0 to t_m. Only in case of Gompertz and Richards function indefinite integrals of those functions do not exist, and the values of definite integrals can be found only by using numerical methods.

Data concerning winter rape *(Brassica napus* L.) plants dry matter accumulation during the spring-summer vegetation were used to calculate the real growth coefficients values. Dry matter of the above-ground parts of rape plants was determined in seven development stages (table 2). The length of spring-summer vegetation period (from spring rosette till pods maturing stage) was 83 days. During that time the mean dry matter of one rape plant increased from 2.2 g to 34.5 g. Position of measuremental points on the chart suggests a typical, i. e. sigmoid course of growth of the analysed plant.

TABLE 2. Dry matter accumulation in winter rape plants (*Brassica napus* L.) during the spring – summer vegetation.

Measurement	Growth stage	Time, t (day)	Dry matter, w (g)
1	Spring rosette	0	2.2
2	Stem formation	13	5.2
3	Inflorescens	27	11.1
4	Flowering	36	19.2
5	Early pods development	50	27.1
6	Pods filling	63	28.8
7	Pods maturing	83	34.5

Winter rape growth was approximated basing on mathematical functions, which are shown in Table 3. The constant linear function coefficients were found by using the least square method. A similar procedure, but after logarithmic transformation, was done for the functions: exponential, Mitscherlich, Gompertz and logistic. The constant coefficients of polynomial of 3rd order was calculated basing on linear multiple regression method, because the regression function was linear in respect to fitting parameters. The constant parameter of difference-differential equation was found numerically by using Raphson-Newton method (Korn and Korn 1968). The constant coefficients of the Richards equation were calculated numerically by using Marquardt's method (Marquardt 1963), after previous initial estimation of the being searched parameters by means of the logistic function (Gregorczyk 1994).

Significance of the applied growth models was verified statistically basing on the correlation coefficient r test or basing on the Fisher-Snedecor F test. Only in case of using the numerical methods (Richards function and the difference-differential one), as an approximation closeness criterion, the

TABLE 1. Review of growth functions and their integrals.

Function	Equation	Indefinite integral (without integration constant)	Definite integral (limit of integration: 0, t_m)
Linear	$w = a + bt$	$at + bt^2/2$	$at_m + bt_m^2/2$
Exponential	$w = a\,e^{kt}$	$a/k\,e^{kt}$	$a/k\,[\exp(kt_m) - 1]$
Polynomial	$w = a + bt + ct^2 + dt^3$	$at + bt^2/2 + ct^3/3 + dt^4/4$	$at_m + bt_m^2/2 + ct_m^3/3 + dt_m^4/4$
Mitscherlich	$w = A(1 - e^{-kt})^2$	$A[t + 2/k\,e^{-kt} - 1/(2k)\,e^{-2kt}]$	$A[t_m + 2/k\exp(-kt_m) - 1/(2k)\exp(-2kt_m) - 3/(2k)]$
Gompertz	$w = A\exp(-b\,e^{-kt})$	—	numerical methods
Logistic	$w = A/(1 + b\,e^{-kt})$	$A[t + 1/k\,\ln(1 + b\,e^{-kt})]$	$A[t_m + 1/k\,\ln[(1 + b\exp(-kt_m))/(1+b)]$
Richards	$w = A\,(1 + b\,e^{-kt})^{1/(1-m)}$	—	numerical methods
Difference-differential	$w = A\,[1 - (1 + kt)\,e^{-kt}]$	$A(t + t\,e^{-kt} + 2/k\,e^{-kt})$	$A[t_m + t_m\exp(-kt_m) + 2/k\exp(-kt_m) - 2/k]$

w – dry matter; t – time of growth; A – asymptotic value of final size; a, b, c, d, k, m – parameters; t_m – final time of growth

TABLE 3. Equations and growth coefficients of winter rape plants (*Brassica napus* L.).

Function	Growth equation of winter rape plants	Value of statistical test	Growth coefficient GC, (g day)	Relative growth coefficient RGC, (g)
Linear	$w = 1.816 + 0.424\,t$	$r^2 = 0.961^{**}$	1611	19.4
Exponential	$w = 3.643\ e^{0.0329\,t}$	$r^2 = 0.857^{**}$	1588	19.1
Polynomial	$w = 1.75 + 0.190\,t + 0.0105\,t^2 - 0.000098\,t^3$	$F = 68.22^{**}$	1637	19.7
Mitscherlich	$w = 34.5\,(1 - e^{-0.0375\,t})^2$	$r^2 = 0.958^{**}$	1564	18.8
Gompertz	$w = 34.5\exp(-3.19\,e^{-0.0468\,t})$	$r^2 = 0.976^{**}$	1623	19.6
Logistic	$w = 34.5 / (1 + 13.7\,e^{-0.0724\,t})$	$r^2 = 0.982^{**}$	1598	19.2
Richards	$w = 34.8\,(1 + 5.49\,e^{-0.0668\,t})^{-1.61}$	$R^2 = 0.991$ 1)	1630	19.6
Difference-differential	$w = 34.5\,[1 - (1 + 0.0521t)\,e^{-0.0521\,t}]$	$R^2 = 0.975$ 1)	1595	19.2

w – dry matter (g); t – time (day); F – Fisher-Snedecor test; r^2 – linear determination coefficient; R^2 – curvilinear determination coefficient; ** – significant value at 1% level; 1) – function parameters were calculated numerically

value of the coefficient of curvilinear determination R^2 was given. All growth models proved to be statistically significant at 1% level. Also R^2 magnitudes which were close to 1 testify to accurate selection of those growth functions. Statistical significance of the linear and exponential function should not base on advisability of usage of the above models for description of those growth processes, in which inflection point is found. Function $w = a + bt$ approximates well only the linear phase of growth and obviously did not show the downward trend of the growth rate in the latter part of the plants vegetation. The exponential function $w = a\,e^{kt}$ circumscribes closely only the initial plant growth stage, when the dry matter increase is proportional to the achieved size of organism. Later on the value of the exponent of the exponential function decreases. It is caused by decrease of the participation of the assimilation tissue in the matter of the whole plant. The lower value of the rectilinear determination coefficient was obtained just for the exponential function ($r^2 = 0.857$), what is a confirmation of the earlier investigations of Ramachandra Prasads et al. (1992).

Table 3 contains also the growth coefficient (GC) and the relative growth coefficient (RGC) values, which were calculated for each growth function. Those coefficients were determined basing on (2) and (4) formulas. Definite integrals of Gompertz and Richards function were calculated out of necessity numerically by using Simpsons method (Korn and Korn 1968).

The growth coefficient (GC) values were included in an interval 1564-1637 g day, and the relative growth coefficient (RGC) values ranged from 18.8 g to 19.7 g. One should pay attention to the relatively little variation of the values of those parameters. It may mean that GC and RGC coefficients are useful for comparison of different forms, cultivars or plant species, after the previous graphical data analysis and the statistical verification of growth curves.

The knowledge of the way of the growth function integrals calculation can be used (basing on equation 3), for determining the mean values of plant dry matter ($\overline{w}$) in any interfacial period. For example underneath was calculated – basing on Richards function and the linear one – the mean value of the winter rape (*Brassica napus* L.) plant dry matter during the period from flowering (36th day of vegetation) to early pods development stage (50th day of vegetation).

$$\overline{w} = \frac{1}{50-36}\int_{36}^{50} (1.816 + 0.424\,t)\,dt = 20.1\ (g)$$

$$\overline{w} = \frac{1}{50-36}\int_{36}^{50} 34.8\,(1 + 5.49\,e^{-0.0668t})^{-1.61}\,dt = 22.4\ (g)$$

Value $\overline{w}$= 22.4 g is closer to the real value, because Richards function approximates the real course of winter rape (*Brassica napus* L.) growth of the analysed plants much closer than the linear function.

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ZASTOSOWANIE RÓŻNYCH FUNKCJI MATEMATYCZNYCH DO OBLICZANIA WSPÓŁCZYNNIKÓW WZROSTU GC I RGC ROŚLIN RZEPAKU OZIMEGO (*BRASSICA NAPUS* L.)

STRESZCZENIE

Przedstawiono metodykę obliczania nowych współczynników charakteryzujących akumulację suchej masy w roślinach. Wykorzystano w tym celu osiem reprezentatywnych funkcji matematycznych oraz dane dotyczące narastania suchej masy roślin rzepaku ozimego (*Brassica napus* L.) w okresie wegetacji wiosenno-letniej. Istotność zastosowanych modeli wzrostu zweryfikowano statystycznymi testami parametrycznymi. W wyniku operacji całkowania analitycznego lub numerycznego wyznaczono dla każdej funkcji współczynnik wzrostu (GC) i względny współczynnik wzrostu (RGC). Określają one odpowiednio potencjał wzrostowy rośliny i średnią wartość suchej masy rośliny w okresie wegetacji. W zależności od rodzaju funkcji matematycznej, wartości współczynnika wzrostu (GC) wahały się w granicach od 1564 g doba do 1637 g doba, a wartości względnego współczynnika wzrostu (RGC) zawierały się w przedziale 18,8-19,7 g.

SŁOWA KLUCZOWE: analiza wzrostu roślin, akumulacja suchej masy, współczynniki wzrostu, rzepak ozimy, *Brassica napus* L.