The logistic function — its application to the description and prognosis of plant growth

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Abstract

A logistic function in the form of $y = A/(1 + be^{-rt})$ was used in this paper to analyse the growth of plants. The first, second and third derivatives of the above equation served as the basis for constructing growth, growth rate and growth acceleration curves. Characteristic points showing the main phases of plant growth were also found on these curves. Very good results were obtained by using the logistic function to describe the accumulation of dry matter by a plant using experimental data from literature. An attempt was also made to forecast growth based on three initial measurements of dry matter. A transcendental equation, solvable by numerical methods, was derived from the three logistic equations. This method was not found to be useful in predicting the final growth due to the large degree of error between the experimental and estimated value.

Key words: plant growth, logistic function, growth prognosis

In the botanical sciences, the term "growth" is usually defined (approached as a phenomenological) as the irreversible increase in the dimensions of a plant due to the autonomic production of organic substances in the course of ontogeny. Growth is a quantitative phenomenon, and as such, is measurable. Depending on the object under study, and the purpose for which the measurements are taken, an indicator of growth may be the increase in dry matter, fresh weight, height, volume, leaf surface, number of cells etc.

Biological literature is full of numerous functions (mathematical models) that describe the process of growth of living organisms (Richards 1959, 1969,

Bertalanffy 1973, Thornley 1976, Savinov et al. 1977, Venus and Causton 1979, Żelawski and Lech 1979, 1980, Sztencel and Żelawski 1984, France and Thornley 1984, Gregorczyk 1986, Causton and Venus 1987). Chronologically speaking, attempts at modelling the growth of animals were undertaken earlier and these models were found to be applicable to the growth of plants also. Growth processes can be presented graphically as an S-shaped curve (a sigmoid curve). Sigmoid curves satisfactorily describe the growth of individual organs, individuals and even entire populations. The general rule in these processes is the initial increase in the growth rate up to the point in which the maximum is reached, followed by a fall in this rate. Growth curves can frequently be described by mathematical functions which are solutions to certain differential equations. From among the numerous growth functions, one called the logistic function (Richards 1969, Vasilyev et al. 1974, Savinov et al. 1976, Seidler and Gregorczyk 1986a, b) has found relatively widespread applications.

The logistic function can be obtained from the general form of the Richards function (Richards 1959, 1969):

$$y = A(1 + be^{-rt})^{1/(1-m)}. \quad (1)$$

At $m = 2$ the above relationship becomes a logistic function in the following form:

$$y = \frac{A}{1 + b \cdot \exp(-rt)}, \quad (2)$$

where: y — a growth size, eg. dry weight,

t — duration of vegetation,

A — maximal end value of growth,

b, r — constant parameters ($b > 0, r > 0$).

A logistic function is differentiable over its entire domain. Its relative growth rate equals:

$$dy/ydt = r(1 - y/A). \quad (3)$$

Figure 1 presents a growth curve, $y(t)$, a growth rate curve, $y'(t)$ and a growth acceleration curve, $y''(t)$. There are three characteristic points in the process of plant growth. These points are marked P_1 , P_i and P_2 on Fig. 1. P_1 is where the maximal acceleration of growth occurs and is the first inflexion of the growth rate curve. P_i is the inflexion point of the sigmoid curve and zero acceleration of growth and the growth rate attains its maximum. P_2 is the point of maximal slow down, that is, maximal negative acceleration of growth and at the same time, is the second inflexion of the growth rate. Points P_1 and P_2 separate growth into three phases: the exponential phase, the linear phase (from P_1 to P_2) and the ageing phase. Points P_1 and P_2 are symmetric to point P_i . Because of the importance of these three critical points, it is important to know their coordinates on the appropriate graphs.

Fig. 1. Graphs of the logistic growth curve (a), the growth velocity curve (b) and the growth acceleration curve (c)

The coordinates of the inflexion point $P_i(t_i, y_i)$ of the logistic function are:

$$t_i = \frac{\ln b}{r}, \quad y_i = A/2. \quad (4)$$

A logistic function is symmetric in respect to its inflexion point and reaches 50% of its maximal value there. The abscissas of the remaining critical points of the growth curve equal:

$$t_1 = t_i - 1.318/r, \quad t_2 = t_i + 1.318/r, \quad (5)$$

while the ordinates of these points may be calculated according to the following formulae:

$$y_1 = A/2 \cdot (1 - \sqrt{1/3}) = 0.211A, \quad y_2 = A/2 \cdot (1 + \sqrt{1/3}) = 0.789A. \quad (6)$$

Equations (6) show that at the first critical point, P_1 , the organism which is subject to changes according to a logistic curve, reaches about 21% of its final size, while in point P_2 — about 79% of its maximum value.

The duration of the linear phase can be calculated from equations (5). This is the so-called significant growth:

$$t_2 - t_1 = 2.636/r. \quad (7)$$

Equation (7) makes it possible to obtain:

$$A = 1.518 \cdot y'_i \cdot (t_2 - t_1). \quad (8)$$

A very important conclusion can be drawn from the above relationship, namely that the final size of a plant or its organ depends only on the maximal growth velocity (y'_i) and the duration of the linear phase.

In order to graph a logistic function on the basis of experimental data, it is necessary to know the final maximal size, A , and to apply the least square method to the linear form of the logistic function:

$$\ln(A/y - 1) = \ln b - rt. \quad (9)$$

In this way, parameters b and r may be solved for, while the number of empirical points may be unlimited.

Theoretically, in order to find the analytical form of the function, it suffices according to Richards (2) to have only three random measured points (t_1, y_1) , (t_2, y_2) , (t_3, y_3) , with $t_1 < t_2 < t_3$ and $y_1 < y_2 < y_3$. A system of three equations, containing the three unknowns: b , r , A , is thus obtained.

$$\begin{cases} y_1 = \frac{A}{1 + b \cdot \exp(-rt_1)} \\ y_2 = \frac{A}{1 + b \cdot \exp(-rt_2)} \\ y_3 = \frac{A}{1 + b \cdot \exp(-rt_3)} \end{cases} \quad (10)$$

Solving the above system of equations makes it possible to obtain the final size, A , and thus may be used to forecast the final size of plants. The system of equations (10) should be reduced to a single equation due to the variable r , introducing at the same time the new parameter, p :

$$p = \frac{(y_3 - y_2)y_1}{(y_2 - y_1)y_3} = \frac{\exp(-rt_2) - \exp(-rt_3)}{\exp(-rt_1) - \exp(-rt_2)}, \quad (11)$$

$p > 0$ and $p < 1$.

Hence:

$$e^{-r(t_3-t_2)} + p \cdot e^{-r(t_1-t_2)} = p + 1. \quad (12)$$

If: $t_3 - t_2 = t_2 - t_1 = \Delta t$, then:

$$e^{-r \cdot \Delta t} + p \cdot e^{r \cdot \Delta t} = p + 1. \quad (13)$$

Substituting $e^{r \cdot \Delta t} = w$, gives the square equation:

$$pw^2 - (p+1)w + 1 = 0. \quad (14)$$

From the two roots of the square equation:

$$w_{1,2} = \frac{p+1 \pm |p-1|}{2p} \quad (15)$$

$w_2 = 1/p$ should be selected (the solution with $w_1 = 1$ corresponds to $r = 0$).
Then:

$$r = \frac{-\ln p}{\Delta t}. \quad (16)$$

The general solution to equation (12) should be analysed now, that is, the case when $t_3 - t_2 \neq t_2 - t_1$.

Let $t_2 - t_1 = a$ and $t_3 - t_2 = \beta$ ($a > 0$, $\beta > 0$),
then:

$$e^{-\beta r} + p \cdot e^{a r} = p + 1. \quad (17)$$

Knowing that $r > 0$ and $p > 0$, the curves of two functions should be drawn: the exponential function derived from the hyperbolic cosine ($\cosh x = 0.5(e^x + e^{-x})$):

$$z = e^{-\beta r} + p \cdot e^{a r} \quad (18)$$

and the linear function:

$$z = p + 1. \quad (19)$$

The abscissas of the intersection point of these two functions correspond to the elements of equation (17). The trivial solution $r_1 = 0$ must be discarded (Fig. 2).

Function (18) has a minimum at point:

$$r_{\min} = \frac{1}{a+\beta} \ln \left(\frac{\beta}{p a} \right) \quad (20)$$

Fig. 2. An auxiliary curve for finding parameter r of the logistic function

For this reason, the second element, r_2 , should be looked for in the range $(r_{\min}, \infty)$. It can be found using numerical methods. Following this, parameters b and A are calculated using the following equations:

$$b = \frac{y_2 - y_1}{y_1 \exp(-rt_1) - y_2 \exp(-rt_2)} \quad (21)$$

$$A = y_1 (1 + b \cdot e^{-rt_1}). \quad (22)$$

The suitability of logistic functions to describe completed growth processes was evaluated on a case taken from literature (Strebeyko 1965) dealing with the course of accumulation of dry matter in one oat plant. On the basis of data from 12 observations, a final dry matter of $A = 24.8$ g was assumed and the least square method was used to calculate the coefficients $b = 240.1$ and $r = 0.1331 \text{ day}^{-1}$. In this way, a specific form of the Richards logistic function was obtained:

$$\hat{y}(t) = \frac{24.8}{1 + 240.1 \cdot \exp(-0.1331t)}. \quad (23)$$

The adjustment of the theoretical curve to the experimental data turned out to be very effective (Table 1, Fig. 3), as is indicated by the values of the straight linear determination coefficient $r^2 = 0.989$ and the curvilinear determination coefficient $R^2 = 0.994$.

This author also attempted to use the logistic function to predict the growth of plants, that is, to determine the final size A based on three measurements made during the initial phase of growth of the analysed oat plants. The calculation scheme presented above was used for this purpose (equations 17-22). The first five measurements (Table 1) were taken for these calculations, and $\binom{5}{3} = 10$ combinations were made from them containing three various observations. Using the Newton-Raphson (Causton and

Fig. 3. A graph of a theoretical curve of accumulation of dry weight of one oat plant as compared with experimental data

Table 1
Experimental (y) and theoretical ($\hat{y}$) values of dry matter of a single oat plant during vegetation (t)

Measure- ment	t (day)	y (g)	$\hat{y}$ (g)
1	15	0.7	0.74
2	20	1.5	1.40
3	25	2.96	2.58
4	30	5.25	4.57
5	35	7.83	7.57
6	40	10.42	11.43
7	45	14.0	15.49
8	50	18.5	18.95
9	55	20.3	21.40
10	60	22.8	22.93
11	65	24.2	23.80
12	75	24.8	24.53

Venus 1987) calculation method, the parameter r was calculated numerically for each trio, followed by parameters b and A . The results of these calculations and the mean values of all of the parameters are presented in Table 2. The average value $A = 12.77$ g obtained in this way significantly deviates from the

experimental value (24.8 g). The error of this estimate is 48.5%, which of course, makes this attempt at applying the logistic function to predict growth processes unsuccessful. It seems that the precision of the prognosis may be increased by using a greater number of measurements, especially if they encompass the maximum growth rate phase (Savinov et al. 1976). It is also difficult not to agree with the opinion of Żelawski and Lech (1980) who hold that functions with an asymptotic end value (the logistic function is such a function) are not suitable for explaining and describing but, at most, may describe plant growth. Of course it should be remembered that the physiological processes of growth of living organisms, especially plants, is a complicated process, dependent on numerous internal and external factors. When constructing a mathematical model of growth the high phenotypic variability of plants, the immense modifying effect of the environment as well as the fact that the final dimensions of plants are not predetermined should be kept in mind. In light of the above it is difficult to expect that a universal model of growth can be developed that is based on biological precepts and suitable for exact forecasting of this process. In spite of this, the problem of modelling growth using mathematical functions remains a contemporary topic of studies.

Table 2
Logistic function parameters obtained on the basis of three observations from the initial five measurements

Observations	r (day ⁻¹)	b	A (g)
1,2,3	0.168	189.1	11.34
1,2,4	0.166	206.7	12.65
1,2,5	0.166	206.4	12.62
1,3,4	0.164	212.7	13.49
1,3,5	0.165	206.0	12.89
1,4,5	0.166	206.4	12.62
2,3,4	0.161	208.6	14.12
2,3,5	0.163	200.7	13.08
2,4,5	0.167	206.8	12.59
3,4,5	0.170	225.7	12.32
Mean values	0.166	206.9	12.77

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Funkcja logistyczna — jej zastosowanie do opisu i prognozowania wzrostu roślin

Streszczenie

W pracy wykorzystano do analizy wzrostu roślin funkcję logistyczną postaci $y = A/(1 + be^{-rt})$. Na podstawie pierwszej, drugiej i trzeciej pochodnej powyższej funkcji wykonano wykresy krzywych wzrostu, szybkości wzrostu i przyspieszenia wzrostu. Znalezione również na tych krzywych charakterystyczne punkty, wyznaczające główne fazy wzrostu roślin. Zastosowano z bardzo dobrym skutkiem krzywą logistyczną do opisu akumulacji suchej masy roślin owsa na podstawie danych doświadczalnych z literatury. Podjęto także próbę prognozowania wzrostu w oparciu o trzy początkowe pomiary wielkości suchej masy. W tym celu z układu trzech równań logistycznych wyprowadzono równanie przestępne, rozwiązywalne metodami numerycznymi. Nie stwierdzono przydatności tej metody do prognozowania końcowej wielkości wzrostu ze względu na znaczny błąd między wartością doświadczalną i oszacowaną.